Due to recent experimental realizations of Bose-Einstein condensation, the theory of ultracold, dilute Bose gases is currently the subject of intensive studies. Moreover, experiments on gases in flat or elongated traps strongly motivate the study of Bose condensation in low dimensional systems, in addition to the undoubtedly interesting three dimensional case.

The mathematical study of the Bose gas goes back to the first quarter of the twentieth century, but the first systematic and semi-rigorous mathematical treatment of Bose-Einstein condensation in (weakly) interacting systems was due to Bogoliubov in 1947 [1]. Such a theory predicts a linear spectrum and provides expressions for the thermodynamic functions which are believed correct in the limit of weak coupling. The 3D formula for the energy per particle, at low density and in the thermodynamic limit, is $e_0(\rho) = 4\pi\mu\rho a$, to leading order in the high dimensional density $\rho^3$. Here $a$ is the scattering length of the two body potential and $\mu = \hbar^2/2m$, with $m$ the mass of the particle. This leading term was proved by Dyson (upper bound) [2] and Lieb-Yngvason (lower bound) [3], whereas the famous second order correction was first derived in [4, 5] and it is known as the Lee-Huang-Yang formula:

$$e_0(\rho) = 4\pi\mu\rho a \left(1 + \frac{128}{15\pi} \sqrt{\rho a^3} + o(\sqrt{\rho a^3})\right).$$

The calculation of the next corrections to Bogoliubov prediction is, so far, an open problem, even if a few recent papers present partial results [6, 7].

With regard to the two dimensional theory, condensation is expected only for zero temperature. The first derivation of the correct asymptotic formula for the ground state energy per particle of a dilute, homogeneous, two-dimensional Bose gas in the thermodynamic limit was postulated as late as 1971 by Schick [8] and was rigorously proved to be correct in [9]. In this case the energy for particle in the thermodynamic limit is $e_0(\rho) \simeq 4\pi\mu\rho |\log \rho a^2|^\frac{1}{2}$, so the energy for N particles is not equal to $N(N-1)/2$ times the energy for two particles in the low density limit, as in the three dimensional case.
In the one-dimensional case, in the presence of repulsive interaction, no condensation is expected, not even at zero temperature. The reference model is the Lieb-Liniger model [10], that is the one-dimensional Bose gas with a pair-wise repulsive $\delta$-function potential. In spite of its integrability, calculating correlation functions in the model is notoriously difficult. Recently a novel approach to compute expectation values in the Lieb-Liniger system has been obtained from a certain non-relativistic limit of the sinh-Gordon model [11], but a complete analysis of the response functions is still lacking.

Apart from very special models [12], until today, nobody has treated the Bose gas at all orders for many-body Hamiltonians with realistic two-body interactions. Lieb, Seiringer and Yngvason [13, 14] have proved complete Bose condensation and superfluidity for 3D and 2D bosons in a trap, but only in the Gross-Pitaevskii limit, in which the density scales with the particle number. A control of the corrections beyond Bogoliubov’s theory is currently beyond reach of rigorous analysis. In fact, all attempts to improve the Bogoliubov approximation encountered the difficulty of a singular perturbation theory plagued by infrared divergences.

However, at weak coupling, at least order by order in (renormalized) perturbation theory, the problem has been solved by Pistolesi et al. [15, 16] and Benfatto [17], who have obtained a strong justification of the generally accepted picture of Bose condensation and superfluid behaviour. In particular they both have proved that the properties of Bose condensation (at zero temperature and weak coupling) for a three dimensional system of bosons, interacting with a repulsive short range potential, can be explained in terms of an asymptotically free renormalization group flow. As main result it has been shown that the two point correlation function has the typical superfluid behaviour at long wavelengths, as generally expected, at least order by order in the running coupling constants. An expression for the sound speed is also obtained, whose leading term (when the coupling goes to zero) coincides with the sound speed in the exactly soluble Bogoliubov model. Pistolesi et al. have investigated the problem for spatial dimensions $1 < d \leq 3$, implementing local Ward identities in a Renormalization Group approach and exploiting an $\epsilon$ expansion with $\epsilon = 3 - d$. In [17] rigorous renormalization group techniques (developed by Gallavotti in the 80’s [18] to study scalar fields) have been used to approach the three-dimensional problem, obtaining “$n!$ bound” on the effective potentials. In two dimension these last techniques have not been implemented yet.

The main purpose of my PhD research project is to develop the strategy presented in [17] to study the order by order estimates for the two dimensional Bose gas. For such a system the analysis of Pistolesi et al. predicts a non trivial fixed point and no anomalous dimensions. Unlike the three dimensional case, in two dimension local Ward identities are crucial for the control of the theory at all orders, in particular to drastically reduce the number of independent running coupling constants. The regularization scheme used by Pistolesi et al. preserves the local gauge invariance required for the Ward identities; however, dimensional regularization is a doubtful method in condensed matter systems, where the possible presence of a lattice induces an ultraviolet momentum cutoff, which explicitly breaks local gauge invariance. Recent works [19, 20] have shown that in low-dimensional systems of interacting fermions (such as Luttinger liquids or graphene) the extra terms appearing in the Ward identities due to the presence of an ultraviolet momentum cutoff are crucial for establishing the infrared behavior of the system: e.g., in one dimension, if one naively neglect them, then no anomalous dimension would emerge, see [21]. For this reason, we expect that non trivial results would derive from the corrections to Ward identities even for bosonic systems. This corrections can be controlled at all orders by implementing the method developed by Benfatto and Mastropietro [21] in the context of fermionic systems. The same ideas can be developed to handle the study of
interacting bosons in one dimension, in order to get a systematic analysis of the correlation functions.

A long term goal for my PhD research is to combine renormalization group ideas with the stationary phase approximations techniques, developed by Balaban et al. [22] for analysing the large distance/infrared behaviour of a system of identical bosons, as the temperature tends to zero. In perspective this would be relevant for a non pertubative analysis of the Bose condensation problem. In fact, the main missing point for a rigorous treatment of BEC is to solve the large field problem. This is not expected to be a trivial generalization of known techniques - as those used to analyse the infrared $\phi^4$ problem in $d = 2, 3$ - since in the Bose gas case one has to use complex Gaussian measures, instead of positive Gaussian measures, and this would introduce new technical problems.

References


Serena Cenatiempo. The theory of ultracold, dilute Bose gases is the subject of intensive studies, driven by new experimental applications, which also motivate the study of Bose-Einstein condensation (BEC) in low dimensions. From the theoretical point of view there are few, quite special, models in which we are able to prove BEC for interacting bosons. With the aim of studying the condensation problem I considered a simplified model for a three and two dimensional system o